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THE STABILITY OF THE JUST SOCIETY

WHY FIXED POINT THEOREMS ARE BESIDE THE POINT

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Political theorists often investigate the attributes of normatively desirable states of affairs. What are the important features of a just society? What makes a democracy desirable? These and similar questions characteristically preoccupy political theorists. In this vein, the question of stability has attracted recurring interest: Can some desirable state of affairs, if realized, persist over time, or are desirable states of affairs bound to degenerate? Alexander Schaefer argues that this focus on the stability of desirable states of affairs—"static political theory" as he calls it—is deeply misguided. The alleged problem is that the question of stability presupposes the existence of "equilibrium" states of affairs. The claim is roughly this: unless a theorist can show that a social-moral system contains an equilibrium state (in a sense to be defined below), there is no point investigating the relative stability of different states within that system. "Before discussing *stability*, the theorist must discuss *existence*."[1] Having exposed this presupposition, Schaefer presents an argument to challenge it, which we reconstruct below. His stated aims are modest. Rather than establish the general nonexistence of social-moral equilibrium states, he "aims to shift the burden of proof": political theorists cannot simply assume that social-moral systems will contain equilibrium states; they "must *prove this*, or at least provide some reason for us to believe it" (1, emphasis in original; also see 9). Yet Schaefer takes this shift to have far-reaching implications: absent an argument that social-moral systems are likely to contain equilibrium states, political theorists "may need to refocus [their] gaze," turning their attention from normatively desirable states of affairs to the "process[es] by which such states arise and are swept away" (1).

Schaefer's conclusion may be correct—perhaps political theorists should spend less effort examining static states of affairs and more time studying

1 Schaefer, "Is Justice a Fixed Point?" 1, emphasis in original; see also 4, 9. Parenthetical page references hereafter refer to the early online version of Schaefer's article.

dynamic social processes. But whatever the merits of this claim, his argument gives us no reason to accept it. The argument fails for two reasons. First, Schaefer's challenge to the existence of social-moral fixed points threatens not only the existence of equilibrium states of affairs but also of robust dynamic processes as he defines that concept. What goes for static political theory goes for his favored "dynamic political theory" too. Second, Schaefer is mistaken about the burden of proof borne by political theorists who are interested in the stability of certain desirable social-moral states. To wit, suppose a theorist claims that a particular state of affairs s is an equilibrium within a larger social-moral system S. Schaefer seems to think that, before this theorist can investigate the stability of s, they must first establish a general existential claim: namely, that we can expect S to contain at least one equilibrium (to be subsequently specified). But that is too strong. Since s is the object of interest, the theorist need only establish *that s is an equilibrium*; if true, this can often be shown without establishing the general existential claim. So Schaefer's argument fails to give static political theorists a reason to rethink the burden of proof they bear; it remains the same as it ever was. Political theorists can continue analyzing social-moral states of affairs, safe in the knowledge that technically formidable fixed point theorems pose no threat to this enterprise.

1. WHAT GOES FOR STATES GOES FOR PROCESSES TOO

Schaefer defines an equilibrium state as a fixed point within a model system (1). Thus, to understand why we should doubt the existence of social-moral equilibria, we must first grasp the concept of a fixed point within a social-moral system (see 5). Suppose we describe social-moral states using a set of (real-valued) variables, of which there are an unspecified number n; thus, each possible social-moral state is identified by a vector of length n, and $\mathbb{R}^n$ defines the space within which possible states are located. Let A be the set of points that contains all possible states. Let a *social-moral system* be a function $f{:}A \rightarrow A$ that takes a social-moral state as an input and returns a social-moral state as an output. A state $x^* \in A$ is a *fixed point* if and only if $f(x^*) = x^*$. Schaefer presents several examples to illustrate this definition. To apply this idea to social-moral systems, Schaefer interprets f as a dynamic transition function: if x describes the state realized by a social-moral system at time t, then $f(x)$ describes the state that emerges from x at $t + 1$ (see 5).[2] Given this interpretation of f, the definition of

2 This temporal quality is added by Schaefer; it is not part of the general definition of f as used in Kakutani's theorem.

a fixed point implies that, if x^* is a fixed point, then, once x^* is realized at some time t, x^* is realized at every time thereafter (2).

We can now briefly restate Schaefer's skeptical argument:

1. Kakutani's fixed-point theorem enumerates four conditions that are jointly sufficient for the existence of an equilibrium state (4–5).
2. For each of Kakutani's four conditions, we can construct a plausible social-moral scenario for which that condition is violated (6–9).
3. "If these counterexamples … sound like plausible descriptions of our own social-moral systems, then we have reason to doubt that there exist any fixed points of justice" (5).
4. "It is reasonable, therefore, to suspect that social-moral systems may also resist equilibration" (1).

Below, we will show why, contrary to premise 3, examples of social-moral systems that violate Kakutani's conditions provide no reason to doubt the existence of social-moral fixed points. For now, we grant this premise for the sake of argument. A key upshot, according to Schaefer, is that political theorists should turn their attention from investigating the stability of desirable social-moral states to investigating the robustness of desirable social-moral processes (10–11). Whereas (equilibrium) states are static, like a snapshot of a social-moral system frozen in time, processes are dynamic, associated with "continual flux" and constant evolution. To capture the distinction, we might think of processes as collections of mechanisms by which social-moral states "arise and are swept away" (1).

A problem arises for Schaefer here: his argument can be used to raise doubts not only about the existence of social-moral equilibrium states but also about the existence of robust social-moral processes. Schafer raises doubts about the former by interpreting a mathematical object, $\mathbb{R}^n$, as defining the space of possible social-moral states. To do this, he interprets the dimensions of $\mathbb{R}^n$ as corresponding to variables we might use to describe the attributes of social-moral states. But we are not required to interpret $\mathbb{R}^n$ in this way, and we might just as well interpret the dimensions of $\mathbb{R}^n$ as corresponding to whatever variables we might use to describe the attributes of social-moral processes. Schaefer's distinction between "*process desiderata* and *state desiderata*" (12) cues us to this alternate interpretation. Just as states can be described as realizing, say, more or less social equality or more or less material welfare, so processes can be described as being better or worse at mitigating violent conflict or providing more or less protection for individuals' rights. So we can think of $A \subseteq \mathbb{R}^n$ as the set of possible social-moral processes. We can also reinterpret f in a similar way: let x describe the operative process at t; then $f(x)$ describes the operative process at $t + 1$. Now, a process $x^* \in A$ is a fixed point if and only if $f(x^*) = x^*$.

Schaefer defines a "robust process" as a process that "maintains some of its qualitative features, even as … society shifts between distinct states" (11). In other words, given a dynamic interpretation of f, a robust process is a fixed point in the space of possible processes: assuming x^* is a fixed point, once x^* becomes operative at t, it will remain operative at $t + 1$.

We can now repurpose Schaefer's argument to raise doubts about the existence of robust social-moral processes, using his counterexamples to Kakutani's conditions as templates for producing our own counterexamples. Just as it "is not difficult to imagine that we could approach full equality or full despotism without ever completely realizing either" (6), so it is not difficult to imagine that we could tinker with a social process so that it continually gets better at mitigating violent conflict or protecting individuals' rights without ever completely eliminating violent conflict or perfectly securing individuals' rights. This gives us an analogue for Schaefer's counterexample to Kakutani's first condition, which requires that the space of possible processes be compact (see 5).[3] Similarly, just as we can imagine how a "public conception of justice might change slowly and continuously, like a stick gradually bending into an arc, until it reaches a critical point where the stick snaps, disrupting a continuous trend that preceded this new state" (7), so we can readily imagine a historical trend in which the social processes of production and wealth accumulation undergo small changes—for example, as inequalities of wealth and political power increase, bequests from parents to children and transfers of resources from the politically powerless to the powerful come to predominate—culminating in a social revolution that sweeps away the old processes of production and replaces them with something entirely different. This gives us an analogue for Schaefer's counterexample to Kakutani's third condition, which requires that f be closed.

We could go on, but we trust we have made our point: if Schaefer has given us reasons to doubt the existence of social-moral equilibrium states, then we can use a slight reinterpretation of the mathematical objects on which his argument depends to generate reasons to doubt the existence of robust social-moral processes. If doubts about the existence of equilibrium states are enough to unsettle the case for doing static political theory, so too doubts about the existence of robust processes must be enough to unsettle the case for doing dynamic political theory.

3 That is, we can think of these hypothetical "perfect" social processes as the unattainable limit points to which sequences of feasible social processes converge, but since they lie outside the set of feasible social processes, that set is not compact.

2. Misplacing the Burden of Proof

No one should be unsettled by the preceding arguments, however, neither ours nor Schaefer's. A theorist who wants to establish that a particular social-moral state is an equilibrium is required to do nothing more (and nothing less) than demonstrate that the state in question is an equilibrium. To take Schaefer's example, if Rawls wants to establish that a society well-ordered by his principles of justice is an equilibrium, then he is required to do nothing more (and nothing less) than show that this state is an equilibrium. Contrary to what Schaefer argues, Rawls is not required to demonstrate that, in general, we can expect (unspecified) equilibrium states to exist.

To consolidate this point, let us consider some examples from applied game theory, which we can treat as a collection of models of limited social-moral systems. Suppose a theorist is studying the factors that foster social cooperation and uses Rousseau's "stag hunt" as a model for the relevant kinds of social interactions.[4] In a "stag hunt," players can either hunt stag together (cooperate) or hunt hare alone (go their separate ways). Suppose our theorist wants to show that the state in which the players cooperate is an equilibrium. Do they first need to demonstrate the general claim that there exists an equilibrium for the stag hunt? Of course not. They need only show that the state in which the players cooperate is an equilibrium. This can be shown directly, without establishing the general existential claim.

Indeed, this is the standard method of argument in applied game theory: the analyst explicitly identifies a particular profile of strategies and directly verifies that it satisfies the conditions for an equilibrium, rather than relying on theorems, like Kakutani's, to first establish that some (unspecified) equilibrium exists and only afterward identifying it explicitly. Very often, the assumptions of those theorems are not met in any case. Consider, for example, a seminal model in political science, the Hotelling-Downs model of electoral competition.[5] In this model, two candidates for political office compete for votes by choosing a "policy platform." The set of possible platforms is represented by some interval on the real number line—say, $[0,1]$—and the candidates can choose any platform within this interval. Each candidate prefers winning to tying and prefers tying to losing. Because the election outcome and thus candidates' payoffs are not continuous functions of the candidates' strategies, the game's best-response correspondence does not satisfy the continuity ("closed

4 Rousseau, "Discourse on the Origins and Foundations of Inequality among Men"; see also Skyrms, *The Stag Hunt and the Evolution of Social Structure*.

5 Hotelling, "Stability in Competition"; Downs, *An Economic Theory of Democracy*.

graph") assumption used in Kakutani's theorem.[6] By Schaefer's reasoning, we should now doubt that this model system has any equilibria and, so, we should hesitate to investigate the properties of any states within this system. Yet it is relatively straightforward to demonstrate that if voters have single-peaked preferences over the set of possible platforms, then the situation in which both candidates choose the median voter's preferred policy platform is an equilibrium; indeed, it is a unique equilibrium.[7] This can be proved directly, without first demonstrating the general claim that there exists an equilibrium for the game.

Schaefer claims that his aim is to "shift the burden of proof" onto static political theorists to "provide some reason to believe" the general claim that "fixed points of justice exist" (5). By this, he seems to mean that they must provide some reason to believe the social-moral system under examination satisfies general conditions ensuring the existence of an equilibrium state. But that is too much to require of a theorist who simply claims that some particular state s is an equilibrium. To be sure, our theorist must show that s is indeed an equilibrium (if that is what they claim). But they have always borne that burden, and Schaefer is wrong to argue that they must bear anything heavier.[8]

One might be persuaded by what we have said here yet struggle to see precisely where Schaefer's reasoning goes astray. In a diagnostic spirit, then, let us think about the general form of his argument:

6 To illustrate, let $m \in (0,1)$ be the location of the median voter's ideal point, and fix $x < m$. For each $n \in \mathbb{N}$, let

$$\hat{x}_n := x - \frac{1}{2n},$$

and

$$x_n := x - \frac{1}{n}.$$

Let

$$\mathbf{x} := (x, x),$$

$$\mathbf{x}_n := (x_n, x_n),$$

and

$$\hat{\mathbf{x}}_n := (\hat{x}_n, \hat{x}_n).$$

Given their payoff functions, for each player, $\hat{x}_n$ is a best response to the other player's choice of x_n for all n. We can see that the sequence $(\mathbf{x}_n, \hat{\mathbf{x}}_n)$ converges to $(\mathbf{x}, \mathbf{x})$, yet choosing x is not a best response when the other player chooses x, as required by the closed-graph assumption.

7 For details, see Gehlbach, *Formal Models of Domestic Politics*, 2–5.

8 If one's goal is merely to establish that a function f has a fixed point, then demonstrating that a particular candidate s (i.e., a point in the domain of f) is a fixed point is as good as any proof that f satisfies general conditions like Kakutani's. Either approach demonstrates that f has a fixed point.

1. Theorist *R* claims that a specific object *s* has property *F*.
2. By an abstruse mathematical theorem, we know that for any object *x* in the relevant domain, *x* has property *F* if *x* satisfies condition *C*.
3. Theorist *R* has given us no reason to believe that *s* satisfies condition *C*, and it is easy to imagine how objects that are similar to *s* might fail to satisfy condition *C*.
4. Thus, we should wonder whether any *x* in the relevant domain has property *F* and, in particular, whether *s* has property *F*, as Theorist *R* claims.

The conclusion does not follow, of course, for two reasons. First, the fact (if it is one) that a few objects that are similar to *s* violate condition *C* does not imply that all objects in the relevant domain violate condition *C* nor even that *s* violates *C*. Second, even if every object in the relevant domain fails to satisfy condition *C*, it could still be that some objects have property *F* since the theorem merely states that *C* is sufficient for *F*.

So theorists who claim that specific social-moral states are equilibria should be unperturbed by Schaefer's argument. If one wants to raise doubts about, say, Rawls's claim that a society well-ordered by his principles of justice (*s*) is an equilibrium (property *F*), then one should engage Rawls's argument for that specific claim rather than raise doubts about the existence of social-moral systems that satisfy Kakutani's conditions (condition *C*). After all, Rawls makes no use of Kakutani's theorem (nor do other theorists), and the assumptions of the theorem are not necessary for the existence of equilibrium states (5). More generally, if one wants to raise doubts about the whole enterprise of analyzing the properties of desirable social-moral states, one needs to do more than sketch a handful of cases that violate a set of conditions that are not necessary for the existence of fixed point equilibria.[9]

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REFERENCES

Downs, Anthony. *An Economic Theory of Democracy*. New York: Harper and Row, 1957.

Gehlbach, Scott. *Formal Models of Domestic Politics*. New York: Cambridge University Press, 2013.

Hotelling, Harold. "Stability in Competition." *Economic Journal* 39, no. 153 (March 1929): 41–57.

Rousseau, Jean-Jacques. "Discourse on the Origins and Foundations of Inequality among Men." In *The Basic Political Writings*. 2nd ed. Translated and edited by Donald A. Cress, 27–92. Indianapolis: Hackett, 2011.

Schaefer, Alexander. "Is Justice a Fixed Point?" *American Journal of Political Science* (forthcoming). https://doi.org/10.1111/ajps.12631.

Skyrms, Brian. *The Stag Hunt and the Evolution of Social Structure*. Cambridge: Cambridge University Press, 2004.